

Math 10 Honours: Section 2.3 Problem Solving Involving Max and Mins

more efficient way.

OR AOS of

$$y = x^2 - 10x$$

$$13: x = \frac{-b}{2a} = 5$$

the "x" coordinate of the vertex, which is the minimum point, is 5. So the numbers are $x=5$ and $x-10=-5$

1. Two numbers have a difference of 10. Their product is a minimum. Determine the numbers

① First number = x

Second number = $x-10$

② product of numbers = $x(x-10) = x^2 - 10x$

product of numbers = y

③ $y = x^2 - 10x$

④ Now, we know the minimum point, so we can get the numbers:

$$x(x-10) = -25$$

$$x^2 - 10x + 25 = 0$$

$$(x-5)^2 = 0$$

$$x = 5$$

first number = 5

second number = $5-10 = -5$

Minimum: $y = x^2 - 10x$

complete the square:

$$y = x^2 - 10x + 25 - 25$$

$$y = (x-5)^2 - 25$$

$$p = 5 \quad q = -25$$

$$\text{vertex} = (p, q) = (5, -25)$$

a minimum point has a "y" value of -25

2. The sum of two natural numbers is 12. Their product is a maximum. Determine the numbers

first natural number = x

second natural number = $12-x$

sum: $x + 12 - x = 12$

AOS: $x = \frac{-b}{2a} = \frac{-12}{-2} = 6$

first number = $x = 6$

second number $12-x = 6$

product of numbers: $x(12-x) = -x^2 + 12x$

product of numbers = y

$$y = -x^2 + 12x$$

• Since the product is the maximum, we can easily get the "x" value of the maximum point (vertex) by getting the equation of the axis of symmetry

3. The sum of two numbers is 60. Their product is a maximum. Determine the numbers.

first number = x

second number = $60-x$

$y =$ the product of numbers

$$(x)(60-x) = y$$

y has to be a maximum

$$-x^2 + 60x = y$$

Axis of Symmetry: $x = \frac{-b}{2a} = \frac{-60}{-2} = 30$

first number = $x = 30$

second number = $60-x = 30$

observation: the maximum product will be given by n^2

The greatest value of the products (MAX) will be given at the vertex. The "x" value that will produce the maximum product will be given by the A.O.S.

4. Two numbers have a difference of 30. The sum of their squares is a minimum. Determine the numbers.

first number = x

second number = $x+30$

A.O.S.: $x = \frac{-b}{2a} = \frac{-60}{4} = -15$

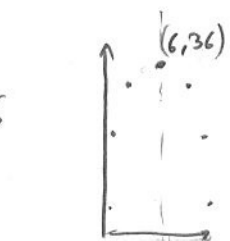
sum of their squares:

$$x^2 + (x+30)^2 = y$$

$$x^2 + x^2 + 60x + 900 = y$$

$$2x^2 + 60x + 900 = y$$

Since "y" is a minimum, we can get the value of "x" that produces the smallest "y" by getting the equation of the A.O.S.



hold on dot b/c they have to be natural numbers. (not 0 or negative)

5. The sum of two numbers is 32. The sum of their squares is a minimum. Determine the numbers.

first number = x $\nearrow x + (32 - x) = 32$
 second number = $32 - x$
 $y =$ sum of their squares.
 Sum of their squares:

$$x^2 + (32 - x)^2 = y$$

$$x^2 + x^2 - 64x + 1024 = y$$

$$2x^2 - 64x + 1024 = y$$

A.O.S. : $x = \frac{-b}{2a} = \frac{-64}{4} = 16$

first number = 16
 second number = 16

For the reason stated in the previous question, we must get the AOS.

6. There is a number such that when you add it to twice its square the sum is minimized. What is this minimum sum?

The number = x
 Twice the sum of the number: $2x^2$
 Their sum = $2x^2 + x$
 their sums = y
 $y = 2x^2 + x$

AOS = $\frac{-b}{2a} = \frac{-1}{4}$ $x = -\frac{1}{4}$

the minimum sum = $-\frac{1}{4} + 2\left(-\frac{1}{4}\right)^2$
 $= -\frac{1}{8}$

Since y has to be minimized, we can get the "x" value of the vertex by getting the AOS.

7. A Broadway musical sells 400 tickets each day at \$30 per ticket. For every increase of \$3.00, they lose 20 sales. i) What should their ticket price be to yield the maximum revenue?

Initial Quantity = $Q = 400$
 Initial Price = $P = \$30$
 Change in Quantity = $\Delta Q = -20$
 Change in Price = $\Delta P = \$3$

$$\frac{Q - 400}{P - 30} = \frac{-20}{3}$$

$$3(Q - 400) = -20(P - 30)$$

$$3Q - 1200 = -20P + 600$$

$$Q = -\frac{20P}{3} + 600$$

Revenue = Quantity $\times$ price
 $= \left(-\frac{20}{3}P + 600\right)P = -\frac{20}{3}P^2 + 600P$
 revenue = $-\frac{20}{3}P^2 + 600P$
 AOS will reveal the "P" value that produces the most revenue
 $P = \frac{-b}{2a} = \frac{-600}{-\frac{40}{3}} = 45$

- ii) What is the maximum revenue? How much more money will they earn compared to their current ticket price?

Optimal price: \$45 $Q = -\frac{20P}{3} + 600$

This will mean that the musical will sell $-\frac{20}{3}(45) + 600 = 300$ tickets.

Max revenue _{produced} with optimal price = $P \times Q = (45)(300) = \$13500$

Current revenue produced = $P \times Q = (30) \times (400) = \$12000$

They will make \$1500 more with the optimal price compared to the current ticket price.

8. A company that charters a boat for tours around Vancouver Island can sell 200 tickets at \$50 each. For every \$10 increase in the ticket price, 5 fewer tickets will be sold.

a. Represent the number of tickets sold as a function of the selling price

Initial Price = $P = 50$

$\Delta P = 10$

$y = mx + b$

Initial Quantity = $Q = 200$

$\Delta Q = -5$

slope: $\frac{50 - P}{200 - Q} = \frac{\Delta P}{\Delta Q} = \frac{10}{-5}$

$$Q(P) = 250 - \frac{1}{2}(P)$$

$$\frac{50 - P}{200 - Q} = \frac{10}{-5}$$

"The quantity of tickets sold is a function of the ticket's price"

$$2000 - 10Q = -250 + 5P$$

$$Q = \frac{P - 450}{-2} = \frac{450 - P}{2}$$

b. Represent the revenue as a function of the selling price

Let $R =$ revenue

Revenue = Quantity $\times$ Price

$$R = QP$$

$$R = \left(\frac{450 - P}{2}\right)(P) = \frac{-P^2 + 450P}{2}$$

$$R = QP$$

$$Q = \frac{450 - P}{2}$$

$$R(P) = -\frac{1}{2}(P^2) + 225P$$

"The revenue is a function of the price"

c. What selling price will provide the maximum revenue? What is the maximum revenue?

$$R(P) = -\frac{1}{2}(P^2) + 225P$$

To get the price that provides the maximum revenue, the equation of the A.O.S. can be used

optimal price = \$225 $R(225) = -\frac{1}{2}(225^2) + 225(225)$

A.O.S.: $x = \frac{-b}{2a} = \frac{-225}{-1} = 225$

Max revenue = $\frac{50625}{2} = \$25312.50$

If the tickets are sold at \$225 each, the maximum revenue of \$25,312.50 will be made.

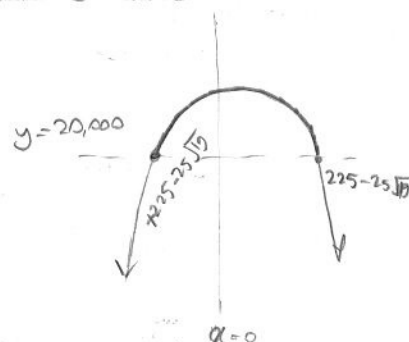
d. What range of price will provide a revenue greater than \$20,000?

$$R(P) = -\frac{1}{2}(P^2) + 225P$$

$$-\frac{1}{2}(P^2) + 225P > 20,000$$

$$P^2 - 450P < -40,000$$

$$P^2 - 450P + 40,000 < 0$$



$$P^2 - 450P + 40,000 = 0$$

$$P = \frac{450 \pm \sqrt{450^2 - 4(40,000)}}{2}$$

$$225 - 25\sqrt{17} < P < 225 + 25\sqrt{17}$$

$$\$121.92 < P < \$328.08$$

$$P = \frac{450 \pm 50\sqrt{17}}{2} = 225 \pm 25\sqrt{17}$$

$$P^2 - 450P + 40,000 < 0$$

9. A company sells its bikes at \$300 each and can sell 70 in a season. For every \$25 increase in the price, the number sold drops by 10.

a. Represent the sales revenue as a function of the price

Initial Price = $P = \$300$
Initial Quantity = $Q = 70$

$\Delta P = \$25$
 $\Delta Q = -10$

$$\frac{70 - Q}{300 - P} = \frac{\Delta Q}{\Delta P} = \frac{-10}{25}$$

$$25(70 - Q) = -10(300 - P)$$

$$350 - 5Q = -600 + 2P \Rightarrow Q = \frac{2P - 950}{-5}$$

Revenue = $R = PQ$
 $R = P\left(\frac{2P - 950}{-5}\right) = -\frac{2}{5}P^2 + 190P$

$$R(P) = -\frac{2}{5}P^2 + 190P$$

b. What price will yield the maximum revenue?

$$R(P) = -\frac{2}{5}P^2 + 190P$$

To get the price that makes the maximum revenue, the Axis of Symmetry should be calculated since the Price at the axis of symmetry will make the maximum amount of revenue.



If the price of each bike is \$237.5, then the max. revenue will be made.

$f(x) = ax^2 + bx + c \Rightarrow R(P) = -\frac{2}{5}P^2 + 190P$
A.O.S.: $x = \frac{-b}{2a}$

$b = 190$ $a = -\frac{2}{5}$ $x = \frac{-b}{2a} = \frac{-190}{(-\frac{2}{5})2} = \frac{190}{\frac{4}{5}} = \frac{475}{2} = 237.5$

c. What range of prices will give a sales revenue that exceed \$18,000?

$-\frac{2}{5}P^2 + 190P > 18,000$

$2P^2 - 950P + 90,000 < 0$

$P^2 - 475P + 45,000 < 0$

$$\frac{475 - 25\sqrt{73}}{2} < P < \frac{475 + 25\sqrt{73}}{2}$$

$$\$130.7 < P < \$344.3$$

$P^2 - 475P + 45,000 = 0$

$aP^2 + bP + c = 0$
 $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$a = 1$
 $b = -475$
 $c = 45,000$

$P = \frac{475 \pm \sqrt{475^2 - 4(45,000)}}{2} = \frac{475 \pm \sqrt{45,625}}{2}$



10. A musical show did a statistical research on the impact of its ticket sold by the ticket price. Through their research, they collected the following data. Using this data, determine what ticket price will yield the maximum revenue.

Ticket Price	tickets sold
\$5.00	416.67
\$6.00	400.00
\$8.00	366.67
\$10.00	333.33
\$12.00	300.00
\$17.00	216.67
\$19.00	183.33

$\Delta P = \$6$

$\Delta Q = -100$

Revenue = $R = PQ$

$R = P\left(\frac{-50P + 1500}{3}\right)$

$R = \frac{-50P^2 + 1500P}{3}$

$R = -\frac{50}{3}P^2 + 500P$

$$\frac{300 - Q}{12 - P} = \frac{\Delta Q}{\Delta P} = \frac{-100}{6}$$

$6(300 - Q) = -100(12 - P)$

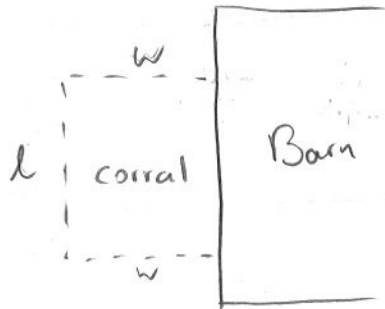
$900 - 3Q = -600 + 50P$

$Q = \frac{-50P + 1500}{3}$

A.O.S.: $x = \frac{-b}{2a} = \frac{-500}{-\frac{100}{3}} = \frac{500 \times 3}{100} = 15$

\$15 as the price of the ticket will yield the maximum value

11. A farmer wants to make a rectangular corral by using his barn wall as one of the sides of the corral. If the farmer has only 60m of fence, what length for the rectangular corral would maximize the area?



$$l + 2w = 60$$

$$l = 60 - 2w$$

$$\text{Area: } lw$$

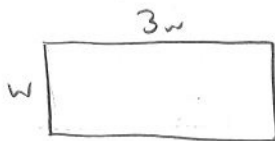
$$(60 - 2w)w = 60w - 2w^2$$

$$y = -2w^2 + 60w$$

$$\text{A.O.S: } x = \frac{-b}{2a} = \frac{-60}{-4} = 15$$

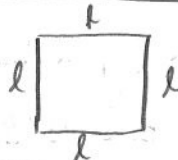
The dimensions of the corral must be 15m by 30m to maximize the area. length = 30m

12. A 50 meter long wire is cut into two separate pieces. One piece is used to make a square and the other into a rectangle, with the length 3 times the width. If the sum of both area is a minimum, find the length of each piece of wire.



$$\text{Area of rectangle} = 3w^2$$

$$\text{Perimeter} = 8w$$



$$\text{Area of square} = l^2$$

$$\text{Perimeter} = 4l$$

$$\text{A.O.S} = \frac{-b}{2a} = \frac{50}{14} = \frac{25}{7}$$

$$w = \frac{25}{7} \quad l = \frac{75}{14}$$

First piece of wire = $8w = \frac{200}{7}$ m
Second piece of wire = $4l = \frac{150}{7}$ m

$$8w + 4l = 50$$

$$4l = 50 - 8w$$

$$l = \frac{25}{2} - 2w$$

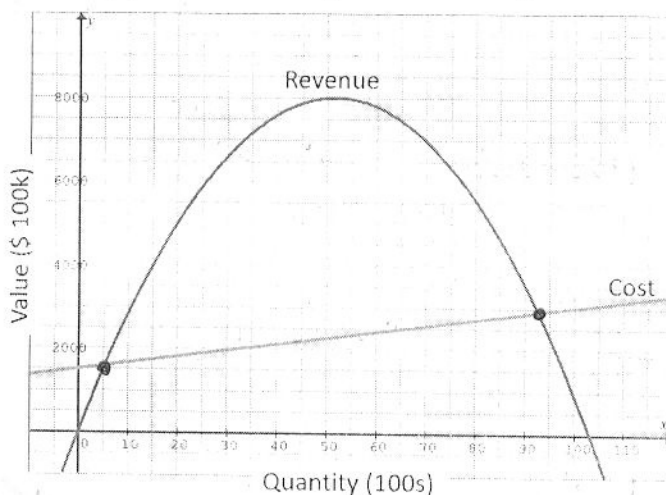
$$y = l^2 + 3w^2$$

$$y = \left(\frac{25}{2} - 2w\right)^2 + 3w^2$$

$$y = 4w^2 - 50w + \frac{625}{4} + 3w^2$$

$$y = 7w^2 - 50w + \frac{625}{4}$$

13. The following graph shows the revenue and cost for producing and selling a certain number of high end watches in a company. Profit is defined as: Profit = Revenue - Cost. Use the graph to answer the following questions: Cost $y = 15x + 1500$, Revenue: $y = -3(x - 51.5)^2 + 8000$



- a) What are the solutions to this system? What do the solution represent?

$$y = 15x + 1500$$

$$y = -3(x - 51.5)^2 + 8000$$

$$\Rightarrow 15x + 1500 = -3(x - 51.5)^2 + 8000$$

$$15x + 1500 = -3x^2 + 309x - 7956.75 + 8000$$

$$3x^2 - 294x + 1456.75 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{294 \pm \sqrt{294^2 - 4 \times 3 \times 1456.75}}{6} = \frac{294 \pm 1689.55}{6}$$

$$x_1 \approx 92.76 \quad x_2 \approx 5.23$$

The solutions represent points on the graph where the revenue made equals to the cost of production. No profit.

b) What quantity will generate the maximum profit? Solve Algebraically

$P = \text{Profit} = \text{Revenue} - \text{Cost}$ $x = Q$ (x variable represents the quantity in this parabola)

$$P = -3(x - 51.5)^2 + 8000 - 15x + 1500$$

$$\text{A.O.S: } x = \frac{-b}{2a} = \frac{-294}{-6} = 49$$

$$P = -3x^2 + 309x - 7956.75 + 8000 - 15x + 1500$$

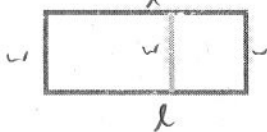
$$P = -3x^2 + 294x + 1543.25$$

c) What would happen to the company financially if they produced over 100,000 watches?

If a quantity of $49 \times (100) = 4900$ watches is produced, the profit will be maximized.

Their Revenue will be lower than their cost of production, so they will lose money.

14. A rectangular area is enclosed by a fence and separated into 2 rectangular regions as shown. With 800m of fencing, what is the maximum area that could be enclosed. Find the dimensions of the enclosed area.



$$\text{area} = lw$$

$$\text{Perimeter} = 2l + 3w = 800$$

$$800 = 2l + 3w$$

$$2l = 800 - 3w$$

$$l = -\frac{3}{2}w + 400$$

$$\text{Maximum Area} = \frac{80000}{3} \text{ m}^2$$

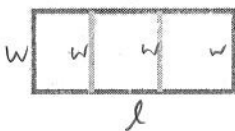
$$\text{Dimensions: } \frac{400}{3} \text{ m by } 200 \text{ m}$$

$$\text{Area} = lw = \left(-\frac{3}{2}w + 400\right)w = -\frac{3}{2}w^2 + 400w$$

$$\text{A.O.S: } x = \frac{-b}{2a} = \frac{-400}{-3} = \frac{400}{3}$$

$$\text{Max Area} = -\frac{3}{2}\left(\frac{400}{3}\right)^2 + 400\left(\frac{400}{3}\right) = \frac{80000}{3} \text{ m}^2$$

15. Suppose the rectangular fence is to be separated into 3 rectangular regions as shown. Again, with 800m of fencing, find the maximum area that could be enclosed. Find the dimensions of the enclosed area.



$$\text{area} = lw$$

$$\text{Perimeter} = 4w + 2l = 800$$

$$2w + l = 400$$

$$l = 400 - 2w$$

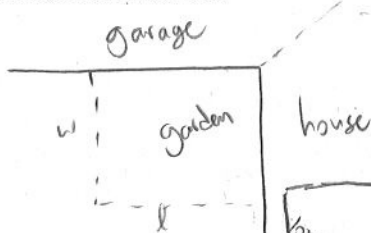
$$\text{Area} = lw = (400 - 2w)w = -2w^2 + 400w$$

$$\text{A.O.S: } x = \frac{-b}{2a} = \frac{-400}{-4} = 100$$

$$w = 100 \text{ m} \quad l = 400 - 2w = 200 \text{ m}$$

$$\text{Maximum Area} = wl = (100)(200) = 20000 \text{ m}^2 \quad \text{Dimensions: } l = 200 \text{ m}, w = 100 \text{ m}$$

16. Bob is going to start a small rectangular garden using his house as one side and his garage as another side. He has 60m of fencing and wants to enclose a maximum area of 450 square metres. How long will the longest side of his fence be?



$$w + l = 60 \text{ metres} \quad w = 60 - l$$

$$lw = \text{Area} = 450$$

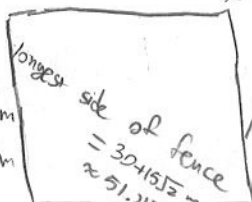
$$lw = l(60 - l) = -l^2 + 60l = 450$$

$$-l^2 + 60l - 450 = 0$$

$$l = \frac{-60 \pm \sqrt{3600 - 1800}}{-2} = \frac{-60 \pm \sqrt{1800}}{-2} = 30 \pm 15\sqrt{2}$$

$$\text{Dimensions: } l = 30 + 15\sqrt{2} \text{ m}$$

$$w = 30 - 15\sqrt{2} \text{ m}$$



Alternate way: $y = -\frac{3}{2}x + 300$

ADS: $x = \frac{b}{2a} = 100$

$y = 150$

Dimensions: 100 by 150

17. A right triangle, with a base of 200m and height of 300m encloses a rectangle as shown. Find the dimensions of the maximum rectangle? Extension: If the height of the triangle is "A" and the base is "B", find the dimensions of the maximum rectangle in terms of "A" and "B".

Dimensions for max square with base of 200m and height of 300m:
 $l = \frac{A}{2} = \frac{200}{2} = 100$ $w = \frac{B}{2} = \frac{300}{2} = 150$

$A_s = \text{Area of Square} = wl$

$A_T = \text{Area of triangle} = wl + \frac{wA - wl}{2} + \frac{lB - wl}{2} = \frac{AB}{2}$

$A_T = wA + lB = AB$

$w = \frac{AB - lB}{A}$ $A_s = wl = \frac{-l^2B + ABw}{A} = \frac{-B}{A} l^2 + Bl$

$A_s = \frac{-B}{A} l^2 + Bl = -\frac{B}{A} \left(l^2 - Al + \left(\frac{A}{2}\right)^2 \right) + \frac{AB}{4} = -\frac{B}{A} \left(l - \frac{A}{2} \right)^2 + \frac{AB}{4}$

vertex = $(x, y) = \left(\frac{A}{2}, \frac{AB}{4}\right)$

Max Area of square:
 $l = \frac{A}{2}, w = \frac{B}{2}$

18. When $f(x) = ax^2 + bx + c$ has a minimum value of "zero", what conditions must be satisfied by a, b , and c

a, b , and c

if not, there will be a max. point

$y = ax^2 + bx + c$

Roots: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

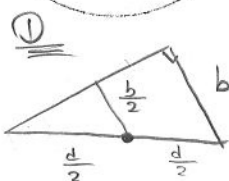
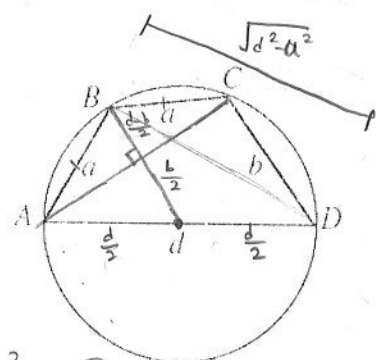
$b^2 - 4ac = 0$

When the discriminant, " $b^2 - 4ac$ ", equals to 0, there will be only one root. If there is one root, the parabola must have a vertex on the x -axis.

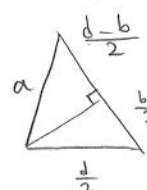
19. Challenge: [euclid] ABCD is a cyclic quadrilateral with side $AD = d$, where "d" is the diameter of the circle. $AB = a$, $BC = a$, and $CD = b$. If "a", "b", and "d" are integers, where $a \neq b$,

a) Prove that "d" can not be a prime number

b) Determine the minimum value of "d"



$\frac{b}{d} = \frac{b/2}{d/2}$



③ $d \neq a^2$

since if $d = a^2$, $\sqrt{d^2 - a^2}$ would be negative, which would be impossible.



so $d = 2a^2$ since that is the only option left that "d" could be equal to.

$d = 2 \times a \times a$

"a" cannot be equal to one since that would result in a non-integer "b" or "d". So $d = 2 \times a \times a$ can never be prime

②

$a^2 - \left(\frac{d-b}{2}\right)^2 = \left(\frac{d}{2}\right)^2 - \left(\frac{b}{2}\right)^2$

$4a^2 + d^2 + 2bd - b^2 = d^2 - b^2$

$2d^2 = 4a^2 + 2bd$

$2d^2 - 2bd = 4a^2$

$d(d - 2b) = 2a^2$

either $d = 2$, $d = 2a$, $d = a^2$, or $d = 2a^2$

$d \neq 2$

since with $d = 2$, "a" & "b" would have to be fractions and not integers

$d \neq 2a$

since with $d = 2a$, $2a$ would have to also be the diameter, but "b" has to be an integer.

b) As seen in the solution for part "a":

$d^2 = 2a^2 + bd$

$d^2 - bd - 2a^2 = 0$

discriminant $b^2 + 8a^2$

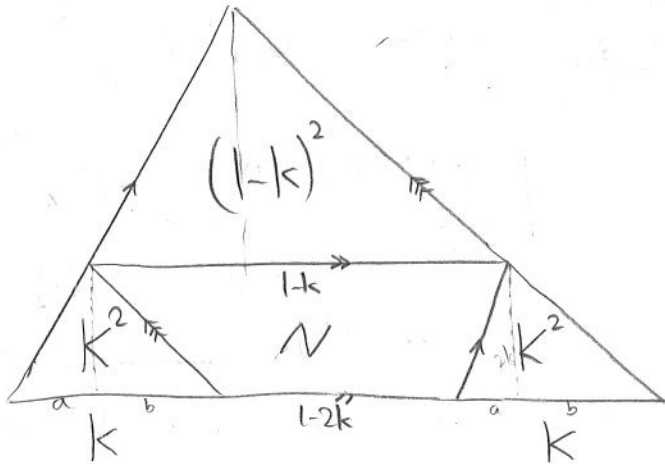
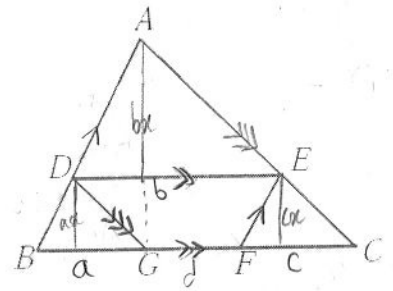
The least value of "a" & "b" that lead to a positive integer "d"

or: $(a, b) = (2, 7)$

Minimum value of "d":

$d = 8$

20. Challenge: In the diagram, the area of $\triangle ABC$ is 1. Trapezoid DEFG is constructed so that "G" is to the left of "F", DE is parallel to BC, EF is parallel to AB and DG is parallel to AC. Determine the maximum possible area of trapezoid DEFG. [Euclid]



$a+b=k$ $\frac{1}{2}$ less assure the base & height are 1.

Area of triangle:

$$k^2 + k^2 + (1-k)^2 + N = 1$$

$$2k^2 + 1 - 2k + k^2 + N = 1$$

$$3k^2 - 2k + N = 0$$

$$N = -3k^2 + 2k$$

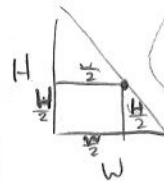
$$N = -3\left(k^2 - \frac{2}{3}k\right)$$

$$N = -3\left[k^2 - \frac{2}{3}k + \left(\frac{1}{3}\right)^2\right] + \frac{1}{3}$$

$$N = -3\left(k - \frac{1}{3}\right)^2 + \frac{1}{3}$$

MAX value of DEFG is $\frac{1}{3}$

Shortcut Br #17.



MAX area of rectangle: $\frac{HW}{4}$

$$y = -\frac{H}{W}x + H$$

$$A = x\left(-\frac{H}{W}x + H\right)$$

$$A = -\frac{H}{W}x^2 + Hx$$

$$A.O.S = \frac{W}{2}$$

$$\text{Max height} = \frac{H}{2}$$

$$\text{Max width} = \frac{W}{2}$$

